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Numerical computation of two-dimensional viscous fingering by a conformal mapping method

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In this paper we numerically compute the long-time evolution of the nonsingular Saffman-Taylor finger (NSTF) with negligible surface tension using a conformal mapping method. Confirming the one-step theory prediction of the NSTF, we resolve one of the Saffman-Taylor riddles successfully. At the same time, we find a successive averaging operator that not only can reduce residual errors in summing up the Fourier series and speed up calculation greatly but also can be easily applied. Its convergence is proved theoretically. [S1063-651X(98)01810-8]

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I. INTRODUCTION

The nonlinear phenomena of incompressible fluid flows have been the subject of many investigations. The nonlinearity of the partial differential equations renders them insolvable. At the same time the most interesting flow properties can be explained only through the nonlinearity. The advent of computers has made it possible to apply suitable numerical techniques in studying fluid flows in the nonlinear regime. Pattern formation and evolution constitute one of the most exciting areas of incompressible fluid flows and viscous fingering in a Hele-Shaw cell [1] is the simplest problem among them. In 1958 Saffman and Taylor [2] showed theoretically that in the absence of surface tension there exists a continuous family of steady-state solutions, i.e., penetrating fingers parametrized by their relative width λ, and they found one of them experimentally. However, several riddles [3,4] remain unresolved and these riddles also exist in the presence of surface tension [5–7]. Recently, Zhao and Yu [8] removed the singularity by a slight modification of the Saffman-Taylor finger and presented a nonsingular Saffman-Taylor finger (NSTF). They then presented a NSTF one-step analysis using a conformal mapping method described by Bensimon [3] and made the following predicitons. (a) If originally $\lambda > 0.5$, the NSTF will become longer and thinner at later times. (b) If originally $\lambda < 0.5$, the NSTF will become shorter and wider at later times. (c) If originally $\lambda = 0.5$, the NSTF will keep its width indefinitely. However, this one-step theory cannot describe the complete shape of the finger over time.

In this paper we compute numerically the long-time evolution of the NSTF with negligible surface tension using a conformal mapping method. The prediction of Ref. [8] is confirmed. Both theoretical and numerical solutions show that the tracing finger always eventually tends to $\lambda = 0.5$ and explain successfully one of Saffman and Taylor's riddles that in the experiment a finger of $\lambda = 0.5$ was observed at each given velocity, but theory could not predict that, since by varying λ one could theoretically obtain fingers of any width.

At the same time, as the Bensimon equation is a highly nonlinear equation, it is numerically unstable and hence tiny roundoff errors, especially the residual errors in summing up the Fourier series, will be magnified quickly. As a result, there always exists a conflict between accuracy and instability. In order to reduce the residual error the number of items N in the Fourier series should be as large as possible. However, large N will introduce high-frequency trigonometric functions, which always make numerical computation unstable and even collapsing. We need to find an approximating operator, which not only reduces residual errors to the greatest extent but also avoids the high-frequency effect, to improve the convergence of the Fourier series. Fejer introduced an operator but still with a low convergence rate. Another approach is the Korbkin operator, which gives a much better approximation but is difficult to construct. Both methods are not suitable for the computation of the NSTF. Here we find that there exists a special operator to approximate the summation of Fourier series by using a successive mean method according to the characteristics of the Fourier series in our computation. It reduces the residual errors to a very low level, making the computation possible and enhancing the calculating speed greatly. Moreover, we prove theoretically that the operator has a better convergence effect and is applied more easily than Fejer and Korbkin operators.

II. EVOLUTION EQUATION

In the notation of Ref. [8], when $t=0$, the NSTF can be written as

$$z = 2(1-\lambda)\ln\tfrac{1}{2}(\zeta + 1 + \epsilon) - \ln\zeta. \qquad (1)$$

The interface is

$$z = 2(1-\lambda)\ln\tfrac{1}{2}(e^{-i\psi} + 1 + \epsilon) + i\psi, \qquad (2)$$

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where the single parameter λ depicts the width of the ratio of the finger to the channel of the Hele-Shaw cell, ψ is the stream function, and $\epsilon=\epsilon(t)$ is a small positive real number varying with time t.

Following Bensimon, we have the finger evolution equation for the mapping

$$\frac{\partial z}{\partial t}=-\zeta\partial_{\zeta}zG(\zeta), \tag{3}$$

where

$$G(\zeta)=\mathcal{A}\{g(\psi)\},\quad g(\psi)=\left[\frac{1}{|\zeta\partial_{\zeta}z(\zeta)|^2}\right]_{\Gamma}. \tag{4}$$

The subscript Γ means that the value of the function in the square brackets is to be taken on the unit circle where $\zeta=e^{-i\psi}$. $\mathcal{A}$ is an operator analytically continuing the real function $g(\psi)$ defined on the circumference of the unit circle as the real part of complex function $G(\zeta)$ in the interior of it. According to the Poisson integral formula [9], if a function $G(\zeta)$, for which the real part on the unit disk $g(\psi)$ can be written as

$$g(\psi)=a_0+\sum_{n=1}^{\infty}(a_ne^{in\psi}+a_n^*e^{-in\psi}), \tag{5}$$

is analytic for $|\zeta|<1$,

$$G(\zeta)=\mathcal{A}\{g(\psi)\}=a_0+2\sum_{n=1}^{\infty}a_nz^n \tag{6}$$

must exist. We rewrite the interface evolution equation as

$$\frac{\partial z}{\partial t}=-A\frac{\partial z}{\partial\psi}, \tag{7}$$

where

$$A=2\sum_{n=1}^{\infty}a_n\sin(n\psi)+ig(\psi), \tag{8a}$$

with

$$a_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{g(\psi)}{2}\cos n\ \psi\, d\psi. \tag{8b}$$

III. NUMERICAL TECHNIQUE

A. Numerical computation of a NSTF

If $z(\psi,t)$ is given, we have $[\zeta\partial_{\zeta}z(\zeta,t)]_{\Gamma}=iz_{\psi}(\psi,t)$. Then we can numerically calculate $z(\psi,t+1)$ from Eq. (7) and obtain the evolution of the finger. However, as expected, Eq. (7) is highly nonlinear and tiny roundoff errors, especially the residual errors in summing up the Fourier series, will be magnified quickly, which makes the calculation very unstable. In order to increase the precision of the summation of the Fourier series and decrease the numerical instability arising from the very-high-frequency noise of trigonometric functions, we have to look for another quickly converging series to approximate A of Eq. (8).

Studying the residual errors

$$\Delta^N=\frac{g(\psi)}{2}-\left(\frac{a_0}{2}+\sum_{n=1}^{N}a_n\cos(n\psi)\right), \tag{9}$$

we find that Δ^N is periodic in the definitive domain of ψ and the larger N is, the smaller the periods. Moreover, the sign of Δ^N alternates as N alternates between odd and even. So assuming

$$\frac{g(\psi)}{2}=\frac{a_0}{2}+\sum_{n=1}^{N}a_n\cos(n\psi)+\Delta^N=\frac{a_0}{2}+\sum_{n=1}^{N-1}a_n\cos(n\psi)$$
$$+\Delta^{N-1}=\frac{a_0}{2}+\sum_{n=1}^{N-2}a_n\cos(n\psi)+\Delta^{N-2}, \tag{10}$$

we easily find

$$\frac{g(\psi)}{2}=\frac{a_0}{2}+\frac{1}{2}a_N\cos(N\psi)+\sum_{n=1}^{N-1}a_n\cos n\psi+\frac{\Delta^N+\Delta^{N-1}}{2} \tag{11}$$

and

$$\frac{g(\psi)}{2}=\frac{a_0}{2}+\frac{1}{4}a_N\cos(N\psi)+\frac{3}{4}a_{N-1}\cos[(N-1)\psi]$$
$$+\sum_{n=1}^{N-2}a_n\cos(n\psi)+\frac{\frac{\Delta^N+\Delta^{N-1}}{2}+\frac{\Delta^{N-1}+\Delta^{N-2}}{2}}{2}. \tag{12}$$

Obviously,

$$\left|\frac{\Delta^N+\Delta^{N-1}}{2}\right|<|\Delta^N|\ \text{ and }\ |\Delta^{N-1}|,$$

$$\left|\frac{\Delta^{N-1}+\Delta^{N-2}}{2}\right|<|\Delta^{N-1}|\ \text{ and }\ |\Delta^{N-2}|,$$

and

$$\left|\frac{(\Delta^N+\Delta^{N-1})/2+(\Delta^{N-1}+\Delta^{N-2})/2}{2}\right|$$
$$\leqslant|\Delta^N|\ \text{ and }\ |\Delta^{N-1}|\ \text{ and }\ |\Delta^{N-2}|,$$

according to the characteristics of Δ's.

This can be repeated until the needed accuracy is obtained. If N is given, we can average the summation from at least one time to at most N times. As we are expecting to choose the minimum N but get the maximum accuracy, we always add N times. Normally, we write

$$\frac{g(\psi)}{2}=\frac{a_0}{2}+\sum_{n=1}^{N}\rho_n^{(N)}a_n\cos n\psi, \tag{13}$$

with successive averaging operator

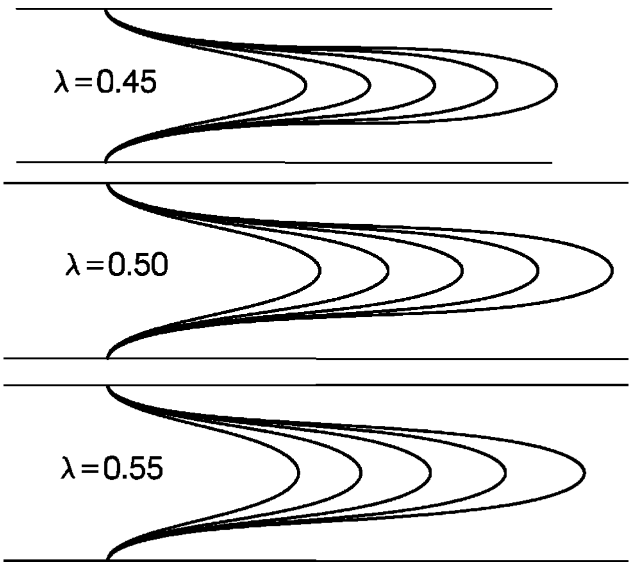

FIG. 1. Stable evolution of the NSTF from $t=0$ to 1.9 with different initial λ and $\epsilon_0=0.3$.

$$\rho_n^{(N)}=\frac{1}{2^N}\sum_{i=n}^{N} C_N^i \quad \text{where} \quad C_N^i=\frac{N!}{i!(N-i)!}.$$

In order to both avoid high-frequency effects and reduce residual error to the greatest extent, we choose $N=7$ and obtain the stable evolution of the NSTF shown in Fig. 1. The prediction that the tracing finger always eventually tends to $\lambda=0.5$ in the one-step analysis is verified.

B. Approximate theory of the successive averaging operator

Although the method described above is specific to a special problem, we can prove theoretically its generality as follows. Assume $g(\psi)\in C[0,2\pi]$, with $\omega(\delta)$ being its continuous modulus. Let the Fourier partial summation of $g(\psi)$ be

$$S_N(g,\psi)=\frac{A_0}{2}+\sum_{n=1}^{N}(A_n\cos n\psi+B_n\sin n\psi), \tag{14}$$

where

$$A_n=\frac{1}{\pi}\int_{-\pi}^{\pi} g(\psi)\cos n\ \psi\ d\psi,$$

$$B_n=\frac{1}{\pi}\int_{-\pi}^{\pi} g(\psi)\sin n\ \psi\ d\psi.$$

Normally, we cannot be certain that $S_N(g,\psi)$ always approximates $g(\psi)$. When the smoothness of $g(\psi)$ is relatively poor, $S_N(g,\psi)$ will not converge $g(\psi)$ or its convergence rate will be very slow.

Let

$$S_N^{(i)}(g,\psi)=\tfrac{1}{2}[S_N^{(i-1)}(g,\psi)+S_{N-1}^{(i-1)}(g,\psi)] \quad i=1,2,\ldots,N, \tag{15}$$

where $S_N^{(0)}(g,\psi)=S_N(g,\psi)$. We introduce a successive averaging operator V as we have used in the computation above:

$$V_N(g,\psi)=S_N^{(N)}(g,\psi) = \frac{A_0}{2}+\sum_{n}^{N}\rho_n^{(N)}(A_n\cos n\psi+B_n\sin n\psi). \tag{16}$$

From

$$\rho_1^{(N)}=1-\frac{1}{2^N}$$

we can see that the course of $\rho_1^{(n)}$ approximating 1 is exponential with order $1/2^N$. This specialty may improve greatly the convergence of the approximation operator.

We rewrite the partial summation $S_N(g,\psi)$ of Fourier series of $g(\psi)$ as

$$S_N(g,\psi)=\frac{1}{\pi}\int_0^{\pi/2}[g(\psi+2t)+g(\psi-2t)]\frac{\sin(2N+1)t}{\sin t}dt \tag{17}$$

so

$$V_N(g,\psi)=\frac{1}{\pi}\int_0^{\pi/2}[g(\psi+2t)+g(\psi-2t)] \times\cos^N t\frac{\sin(N+1)t}{\sin t}dt. \tag{18}$$

Now we define the residual error $E_N(g,\psi)=V_N(g,\psi)-g(\psi)$; then

$$E_N(g,\psi)=\frac{1}{\pi}\int_0^{\pi/2}[g(\psi+2t)+g(\psi-2t)-2g(\psi)] \times\cos^N t\frac{\sin(N+1)t}{\sin t}dt. \tag{19}$$

Assuming $I_N=\int_0^{\pi/2}\cos^N\psi\,d\psi$, we have $I_N^2\leqslant\pi/2N$. Introducing an arbitrary positive number δ, we have

$$\int_0^{\delta/2}\cos^N\psi\ d\psi\leqslant\sin(\delta/2)\sqrt{\frac{\pi}{2N}}. \tag{20}$$

Let

$$\Delta_1=\frac{1}{\pi}\int_0^{\delta/2}[g(\psi+2t)+g(\psi-2t)-2g(\psi)] \times\cos^N t\frac{\sin(N+1)t}{\sin t}dt, \tag{21a}$$

$$\Delta_2=\frac{1}{\pi}\int_{\delta/2}^{\pi/2}[g(\psi+2t)+g(\psi-2t)-2g(\psi)] \times\cos^N t\frac{\sin(N+1)t}{\sin t}dt. \tag{21b}$$

If $\omega(\delta)$ is the continuous modulus of the function $g(\psi)$, we have

$$|\Delta_1| \leqslant \frac{2}{\pi}\omega(\delta)(N+1)\sin(\delta/2)\sqrt{\frac{\pi}{2N}} \leqslant C_1\delta\sqrt{N}\omega(\delta), \tag{22a}$$

with C_1 a positive constant that is independent of δ, n, and $g(\psi)$, and

$$|\Delta_2| \leqslant \frac{2\omega(\pi)}{\pi\sin(\delta/2)}\cos^{N/2}(\delta/2)\int_{\delta/2}^{\pi/2}\cos^{N/2}t\, dt \leqslant \frac{C_2}{\delta\sqrt{N}}(1-\delta^2)^{N/4}, \tag{22b}$$

with C_2 a positive constant that is independent of δ and n. Thus

$$|E_N(g,\psi)| \leqslant C_1\delta\sqrt{N}\omega(\delta) + \frac{C_2}{\delta\sqrt{N}}(1-\delta^2)^{N/4}. \tag{23}$$

Supposing $\theta = \delta\sqrt{N}\omega^{\beta}(\delta)$ is a given small positive constant, where β is another given constant with $0<\beta<1/2$, we have

$$|E_N(g,\psi)| \leqslant C_1\theta\omega^{1-\beta}(\delta) + \frac{C_2}{\theta}\omega^{\beta}(\delta)\exp\left[-\frac{\theta^2}{4\omega^{2\beta}(\delta)}\right]. \tag{24}$$

Obviously, we get from Eq. (24) $|E_N(g,\psi)| \to 0$ when $\delta \to 0$ (or $N\to\infty$).

Moreover, we can see that the residual error can be divided into two parts: Δ_1, determined by the continuous modulus of $g(\psi)$ with a given small coefficient, and Δ_2, attenuating exponentially when $\delta\to 0$ (or $N\to\infty$). Therefore, $V_N(g,\psi)\to g(\psi)$ very quickly. Our numerical computation has verified this result.

IV. CONCLUSION

By modifying the Saffman-Taylor finger into the NSTF, our theoretical and numerical work has the following merits in the field.

First, we vest a finger with real experimental background and make the comparison between theory, numerical analysis, and experiments more meaningful. Here one can answer why Saffman and Taylor could find just one finger with $\lambda = 0.5$ in their experiment in 1958.

Second, the NSTF makes it possible to calculate the evolution of a single viscous finger, which provides more models for computing numerically the evolution, competition, and ramification of straight, parallel fingers as well as radial fingers.

Third, we present a fast, effective algorithm that has much better convergence effects. Summing a finite number of items in an infinite series expanded from a function is a normal but useful computational method. Successive averaging operators can be put into practice easily. We just substitute $N=7$ for ∞ of Eq. (8a) and iterate more than 20 000 steps stably, which makes us trace the evolution of the finger successfully. As a comparison, we use the Fejer operator instead of the successive averaging operator without changing any other parameters. The computation shows that less than 100 steps when $N=7$ and less than 1000 steps when $N=70$ can be iterated before the unstable patterns emerge. The generality proof demonstrates the scientific merit of the successive averaging operator further.

On the basis of the NSTF and the successive averaging operator, we are studying further work dealing with surface tension and stability to resolve another riddle.

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